UDC 539. 3

SETTLING OF A PLANE STAMP ACTING ON AN ORTHOTROPIC HALF-SPACE

PMM Vol. 42, № 1, 1978, pp. 189-192

V. A. SVEKLO and N. S. TORUBAROVA
(Kaliningrad)

(Received October 19, 1976)

Settling of a stamp is studied for a particular case of an orthotropic body. Certain supplementary conditions for the existence of a solution are elucidated. Settling of a stamp is computed for concrete anisotropic bodies, and the influence of rotation of the stamp axes taken into account.

1. Derivation of basic formulas. The stress-strain state of an orthotropic half-space acted upon by a flat stamp elliptic in the plane, is defined by the load function

$$\Psi(\Omega_k) = P(8\pi^2 \sqrt{ab})^{-1} \ln [(\Omega_k - \sqrt{ab})(\Omega_k + \sqrt{ab})^{-1}] \tag{1.1}$$

$$\Omega_k = (\xi + \nu_k z)\Delta^{-1}, \quad \Delta = (a^2\alpha^2 + b^2\beta^2)(ab)^{-1}$$

$$\xi = \alpha x + \beta y, \quad \alpha = \cos\theta, \quad \beta = \sin\theta$$

Here P denotes the compressive force applied to the stamp at the center of the ellipse, a and b are the ellipse semiaxes and ν_k are the roots of the characteristic equation (1.2) of [1]. We assume that the axes of the ellipse coincide with the elastic symmetry axes of the medium, i.e. the boundary plane $z = 0$ coincides with one of the elastic symmetry planes of the body.

The function (1.1) corresponds to the following stress distribution under the stamp:

$$\sigma_z(x, y, 0) = P(2\pi ab)^{-1}(1 - x^2/a^2 - y^2/b^2)^{-1/2} \tag{1.2}$$

and outside the stamp we have $\sigma_z(x, y, 0) = 0$.

The elastic displacements of the points of the medium under the stamp, orthogonal to the boundary, are all equal and given by the formula

$$w(x, y, 0) = P(2\pi\sqrt{ab})^{-1}\left\langle \sum_{k=1}^{3} \operatorname{Re} i\Delta_k^{(3)}\Delta_k(\Delta_0\Delta)^{-1}\right\rangle \tag{1.3}$$

The values of $\Delta_k^{(3)}, \Delta_k, \Delta_0$ are shown in [1]. Here and henceforth the angular brackets will denote integration in θ from 0 to 2π. The formula (1.3) defines the settling of the stamp; the settling depends essentially on the material of the medium.

In what follows, we shall restrict ourselves to considering a particular form of an orthotropic body in which the elastic constants satisfy the conditions

$$B = A, \quad M = L, \quad G = F$$

In this case we have [2]

$$w(x, y, 0) = P(2\pi\sqrt{ab})^{-1}\langle \operatorname{Re} i\Delta_1^* (\Delta_0^*\Delta)^{-1}\rangle \tag{1.4}$$

$$\Delta_1^* = -(L + F)(CL)^{-1}D(\nu_1 + \nu_2)(\nu_2 + \nu_3)(\nu_3 + \nu_1)$$

$$\Delta_0^* = -(L + F)(CL)^{-1}D\Delta_0^{**}$$

$$\Delta_0^{**} = (AC - F^2)C^{-1} - 2K_0\alpha^2\beta^2 + (LF)^{-1}\{N(AC - F^2) + K_0[(A + H)C - 2F^2]\alpha^2\beta^2\}F^{1/2}(DC)^{-1/2}m$$
$$D = (LF)^{-1}[AN + K_0(A + H)\alpha^2\beta^2]$$
$$K_0 = A - 2N - H, \quad m = \operatorname{Jm}(\nu_1 + \nu_2 + \nu_3)$$
$$HC - F^2 > 0, \quad A > H$$

(misprints in the expressions Δ_1^* and Δ_0^* made in [2] have been eliminated).

The characteristic equation (1.2) of [1] can be written conveniently in the form

$$\nu^6 + A_2\nu^4 + A_4\nu^2 + A_6 = 0 \tag{1.5}$$
$$A_2 = (CL)^{-1}=[AC + L^2 - (L + F)^2 + CN]$$
$$A_4 = (CL^2)^{-1}\{AL^2 + N[AC + L^2 - (L + F)^2] + [C(A + H) - 2(L + F)^2]K_0\alpha^2\beta^2\}, \quad A_6 = FC^{-1}D$$

When the inequalities shown above hold, the quantities A_j are positive for all $\theta \in (0, 2\pi)$, and this implies that Eq. (1.5) has no real roots. This agrees with the requirement that the system of elastic equilibrium equations be fully elliptic.

From (1.4) it is clear that $w(x, y, 0)$ is a symmetric function of any three different roots of (1.5). We can assume without loss of generality that their imaginary parts m_k $(k = 1, 2, 3)$ are positive, i.e. $m = m_1 + m_2 + m_3 > 0$ for all θ in the interval shown above.

From (1.5) we can derive the following algebraic equation for m:

$$f(m) = (m^2 - A_2)^2 - 8\sqrt{A_6}m - 4A_4 = 0 \tag{1.6}$$

On the other hand we have

$$(\nu_1 + \nu_2)(\nu_2 + \nu_3)(\nu_3 + \nu_1) = -{}^1/_8 if'(m)$$

hence we can write (1.4) in the form

$$w(x, y, 0) = P(16\pi\sqrt{ab})^{-1}\langle f'(m)(\Delta_0^{**}\Delta)^{-1}\rangle \tag{1.7}$$

Since the quantity w must be positive and $\Delta_0^{**}(m) > 0$ for all θ, the problem of existence and uniqueness of a solution of the problem in question depends, in particular, on the following conditions being fulfilled: (a) positive roots of (1.6) exist; (b) a unique positive root m of this equation exists for which $f'(m) > 0$.

Let us investigate the roots of Eq. (1.6). The following cases are possible at various values of θ from the interval $(0, 2\pi)$:

$$A_2^2 - 4A_4 \leqslant 0, \quad A_2^2 - 4A_4 > 0$$

We have $f(\pm\infty) > 0$, $f(\sqrt{A_2}) < 0$. In the first case $f(0) \leqslant 0$, and in the second case $f(0) > 0$. Since $f'(m)$ has a single positive root lying to the right of the point $\sqrt{A_2}$ and the points $\sqrt{{}^1/_3 A_2}$ are points of inflection, it follows that in the first case we have only a single positive root of (1.6), while in the second case we have two roots. In the latter case $f'(m) > 0$ only for the larger of those two roots, therefore the conditions given above hold for both cases.

Differentiating the identity (1.6) with respect to θ we obtain

$$m_\theta' = K_0 T_0 [f'(m)]^{-1} \sin 4\theta$$

$$T_0 = (2LC\sqrt{A_6})^{-1}\{(A+H)m + \sqrt{A_6}L^{-1}[C(A+H) - 2(L+F)^2]\}$$

Since $f'(m) \neq 0$, m attains an extremum when $\theta = k\pi/4$ $(k = 1, 2, \ldots)$. If $T_0 > 0$ (the condition which holds for all real media referred to in [3], then the sign of the second derivative of m with respect to θ is determined by the sign of the expression $K_0 \cos 4\theta$. If $K_0 > 0$ $(K_0 < 0)$, then m attains a minimum (maximum) at $\theta = 0, \pi/2$ and a maximum (minimum) at $\theta = \pi/4$ (we only need to consider the first quarter).

When $\theta = 0, \pi/2$ the roots ν_k of (1.5) are obtained in the following explicit form:

$$\nu_{1,2}^2 = (2LC)^{-1}\{[AC - F(F+2L)] \mp \sqrt{[AC-(F+2L)^2](AC-F^2)}\}$$

$$\nu_3^2 = -NL^{-1}$$

The same result is obtained by setting formally in (1.5) $K_0 = 0$, i.e. by introducing a "model" transversely isotropic medium for the material in question. For this medium the quantity m will be constant and described, in the polar m, θ coordinates, by a circle the radius of which is m when $\theta = 0, \pi/2$. If $K_0 > 0$ for the given material, then the curve representing the dependence of m on θ will be found outside this circle, and inside when $K_0 < 0$. The first medium is represented by e.g. sylvite, fluorspar, rock salt and cubic pyrites, while topaz and barytes can serve (after some averaging of the elastic constants) as examples of media for which $K_0 < 0$.

Table 1

	$\theta° = 0$	15	30	45
1	3.49059	3.49059	3.49059	3.49059
2	3.01441	3.00200	2.97596	2.96226
3	3.41205	3.39110	3.34655	3.32274
4	3.36718	3.39839	3.45586	3.48252
5	3.43527	3.47047	3.53457	3.56404
6	3.20477	3.21953	3.24774	3.26124
7	3.25234	3.27046	3.30473	3.32102

For a transversely isotropic body $(K_0 \equiv 0)$ $\Delta_0^{**}(m_0)$ and $f'(m_0)$ are all constant, and the settling of a plane stamp of elliptical crossection is determined by the formula

$$w(x, y, 0) = P(16\pi\sqrt{ab})^{-1} f'(m_0)\Delta_0^{**}(m_0)\langle\Delta^{-1}\rangle$$

where the elliptic integral can be found from tables. In the case of a circular stamp the result can be obtained in terms of the elementary functions.

If $K_0 \neq 0$, the largest root of (1.6) can be found for any value of θ using the method of consecutive approximations and the equation

$$m^2 = A_2 + 2\sqrt{A_4 + 2\sqrt{A_6}m}$$

which yield an increasing bounded sequence converging to m. For an isotropic medium we have $A_2 = A_4 = 3$, $A_6 = 1$, $m = 3$.

Table 1 gives m for certain values of θ for beryl (1), topaz (2), barytes (3), cubic pyrites (4), fluorspar (5), rock salt (6) and sylvite (7). The data for topaz and barytes were obtained after averaging their elastic constants (neither material belongs to the class in question [3]). The following values were accepted

for topaz $A = 3215$, $C = 3000$, $H = 1280$, $F = 880$, $L = 1225$, $N = 1330$;
for barytes $A = 854$, $C = 1074$, $H = 468$, $F = 274$. $L = 208$, $N = 283$.

The settling of the stamp was computed for the anisotropic media shown above, using the formula

$$w = P\,(64\pi a\cdot 10^6)^{-1} T\,(e), \quad e = b/a$$

$$T\,(e) = \langle f'\,(m)\,(\Delta_0^{**}\Delta^\circ)^{-1}\rangle, \quad \Delta^\circ = (\alpha^2 + e^2\beta^2)^{1/2}$$

The values of $T\,(e)\cdot 10^5$ are given in Table 2 (where the media are numbered as in Table 1).

It should be noted that for the model media introduced above corresponding values of $T\,(e)$ differ from those in Table 2 only in the fourth decimal place. It therefore appears that in practice it is sufficient to carry out the basic computation of the settling of the stamp.

2. Settling of a plane elliptic stamp the axes of which are inclined relative to the elastic symmetry axes of the material. Let the major semiaxis of the loading ellipse be inclined to the x-axis at the angle γ_0. Then the complex solutions of the elastic equilibrium equations can be constructed in the form

$$u_j\,(x, y, z) = \sum_{k=1}^{3} \langle \operatorname{Re} u_j^*(\Omega_{1k})\,\Delta^{-2}\rangle \quad (j = 1, 2, 3) \tag{2.1}$$

$$\Omega_{1k} = (\xi_1 + \nu_{1k} z)\Delta^{-1}, \quad \xi_1 = x_1\alpha + y_1\beta$$

$$x_1 = x\cos\gamma_0 + y\sin\gamma_0, \quad y_1 = -x\sin\gamma_0 + y\cos\gamma_0$$

where we introduce new notation for the elastic displacements of the points of the medium, and in particular $u_3\,(x, y, z) = w\,(x, y, z)$. The function $u_j^*\,(\Omega_{1k})$ represents the functions $u_j\,(x, y, z)$ in the above sense. We have

$$\xi_1 = x\alpha_1 + y\beta_1, \quad \alpha_1 = \cos\,(\theta + \gamma_0), \quad \beta_1 = \sin\,(\theta + \gamma_0)$$

which together with (2.1), imply that the settling of the stamp is determined in this case by (1.7) where α and β in the expressions for $f'\,(m)$ and Δ_0^{**} are replaced by α_1 and β_1. It is evident that for an isotropic medium and a transversely isotropic body the inclination of the axes of the stamp does not yield a new value for its settling when the z-axis coincides with the elastic symmetry axis.

Table 2

	$e = 1$	$1/2$	$1/4$	$1/100$
1	1224	1680	1919	2498
2	861	1829	1352	1762
3	3195	4387	5006	6520
4	838	1149	1319	1712
5	1997	2741	3128	4067
6	6321	8677	9800	12877
7	10545	14473	16516	21476

Table 3

	$\gamma_0^\circ = 5$	15	30	45
1	1680	1680	1680	1680
2	1185	1319	1406	1435
3	4386	4881	5203	5309
4	1156	1283	1368	1397
5	2740	3046	3247	3314
6	8673	9644	10280	10492
7	14469	16086	17148	17502

We also see that the settling of the stamp is identical when $\gamma_0 = 0, \pi/2$. It can be shown that the settling is extremal at the above values of γ_0, as well as at $\gamma_0 = \pi/4$.

Table 3 gives the values of $T_1 \cdot 10^5$ for certain values of γ_0 and $e = {}^1/_2$ (the media are numbered as in Table 1).

The results obtained indicate that maximum settling occurs for all media when $\gamma_0 = \pi/4$.

REFERENCES

1. Sveklo, V.A., The action of a stamp on an elastic half-space. PMM, Vol. 34, No.1, 1970.

2. Sveklo, V.A., Hertz problem on compression of anisotropic bodies. PMM, Vol. 38, No.6, 1974.

3. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity. N.Y. Dover, 1944.

Translated by L.K.